Reducible to Exact

Type - I

Ex. 1. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$M = x^2y - 2xy^2$, $N = -(x^3 - 3x^2y)$

$\Rightarrow \frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = -3x^2 + 6xy$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

It is an homogenous eq^n.

Now, $Mx + Ny = \cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2 = x^2y^2$

$\Rightarrow I.F = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

$\Rightarrow \frac{1}{x^2y^2}\left\{(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy\right\} = 0$

$= \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

Now, $M' = \frac{1}{y} - \frac{2}{x}$, $N' = \frac{-x}{y^2} + \frac{3}{y}$

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N'}{\partial x} = \frac{-1}{y^2}$

$\Rightarrow$ The equation is exact.

Hence, the solution is

$\int_{y\text{-Constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = 0$

$\Rightarrow \frac{x}{y} - 2\log x + 3\log y = c$

$\Rightarrow \frac{x}{y} + \log\frac{y^3}{x^2} = c$ Ans//

Type - II

2. $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Here, $M = xy^2 + 2x^2y^3$, $N = x^2y - x^3y^2$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2, \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now, $Mx - Ny = \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3 = 3x^3y^3$

Now, I.F $= \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$

$$\Rightarrow \frac{1}{3x^3y^3}\left\{\cancel{y}(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy\right\} = 0$$

$$\Rightarrow \frac{1}{3}\left\{\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{\cancel{3}xy^2} - \frac{1}{y}\right)dy\right\} = 0$$

Here, $M' = \frac{1}{x^2y} + \frac{2}{x}$, $N' = \frac{1}{xy^2} - \frac{1}{y}$

$$\Rightarrow \frac{\partial M'}{\partial y} = \frac{-1}{x^2y^2}, \quad \frac{\partial N'}{\partial x} = \frac{-1}{x^2y^2}$$

Here, $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$\Rightarrow$ The given equation is exact.

Hence, the solution is

$$\underset{y\text{-constant}}{\int}\left(\frac{1}{x^2 y} + \frac{2}{x}\right) dx + \int -\frac{1}{y}\, dy$$

$$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = c$$

$$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = c \text{ Ans}$$

Type - III

Q. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$

Here, $M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2$, $N = \frac{1}{4}(x + xy^2)$

$\Rightarrow \frac{\partial M}{\partial y} = 1 + y^2 + x$, $\frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + y^2 - \frac{1}{4} - \frac{y^2}{4} = \frac{3}{4} + \frac{3y^2}{4} = \frac{3}{4}(1+y^2)$

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x(1+y^2)} \cdot \frac{3}{4}(1+y^2) = \frac{3}{x}$

Now, I.F $= e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} = e^{3\int \frac{dx}{x}} = e^{3\log x} = e^{\log x^3} = x^3$

Hence, eq^n becomes

$\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \frac{1}{4}(x^4 + x^4y^2)\,dy = 0$

Here, $M' = x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}$, $N' = \frac{x^4}{4} + \frac{x^4y^2}{4}$

$\cancel{\frac{\partial M}{\partial y} = 3x^3 + 8x^2y^3}$ $\quad \cancel{\frac{\partial N}{\partial x} = x^3 + \frac{4x^3y^2}{4} = x}$

$\frac{\partial M'}{\partial y} = x^3 + x^3y^2$, $\frac{\partial N'}{\partial x} = x^3 + x^3y^3$

Here, $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$\Rightarrow$ The given equation is exact.

Hence, the solution is

$$\int_{y\text{-Constant}} \left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \int 0\,dy = 0$$

$$\Rightarrow \frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = K$$

$$\Rightarrow 3x^4y + x^4y^3 + x^6 = 12K$$

$$\Rightarrow x^4y(3y + y^3) + x^6 = C \quad \text{Ans}$$

Type – IV

6. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

Here, $M = 3x^2y^4 + 2xy$, $N = 2x^3y^3 - x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \qquad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6x^2y^3 - 4x$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y\cancel{(3x^2y^3 + 2x)}} \cdot -2\cancel{(3x^2y^3 + 2x)} = \frac{-2}{y}$$

Now, $$I.F = e^{-2\int \frac{dy}{y}} = e^{-2\log y} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

Hence, the equation becomes

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

Here,

$$M' = 3x^2y^2 + \frac{2x}{y}, \quad N' = 2x^3y - \frac{x^2}{y^2}$$

$$\frac{\partial M'}{\partial y} = 6x^2y - \frac{2x}{y^2}, \quad \frac{\partial N'}{\partial x} = 6x^2y - \frac{2x}{y^2}$$

Here, $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$\Rightarrow$ The given equation is exact.

Hence, the solution is

$$\underset{y\text{-constant}}{\int}\left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0 . dy = c$$

$$\Rightarrow x^3y^2 + \frac{x^2}{y} = c \text{ Ans}$$